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MRC Technical Summary Report #2320

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February 1982

(Received January 20, 1982)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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C. de Boor[1] and K. Höllig[1,2]

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ABSTRACT

We study multivariate B-splines M obtained as "shadows" of parallelepipeds, and spaces spanned by their translates $S = \operatorname{span}_{j\in\mathbb{Z}^m} M(\cdot - j)$. Recurrence relations for M are obtained and a necessary condition for the stability of the B-spline basis is given. We further determine the polynomials contained in S and the optimal degree of approximation from S.

AMS (MOS) Subject Classifications: 41A15, 41A63, 41A25

Key Words: B-splines, multivariate, spline functions, degree of approximation

Work Unit Number 3 (Numerical Analysis and Computer Science)

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[1]Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

[2]This material is based upon work supported in part by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

SIGNIFICANCE AND EXPLANATION

Local support bases for piecewise polynomial spaces are important for applications such as finite element methods, data fitting etc. In [BH_1] a general construction principle for such "B-splines" was described. A special case are the so called box-splines. They have a particularly regular discontinuity pattern and coincide in special cases with standard finite elements.

It is hoped that using translates of box-splines will lead, at least in two variables, to a unified theory for piecewise polynomial functions on regular meshes.

This note is a first attempt in this direction and deals with basic approximation properties of translates of one box-spline such as stability, degree of approximation etc.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

B-SPLINES FROM PARALLELEPIPEDS

C. de Boor[1] and K. Höllig[1,2]

0. Introduction. Following $[BH_1]$, we define the **B-spline** M_B as the **m-shadow** of the polyhedral convex body $B \subseteq \mathbb{R}^n$, i.e., as the distribution on $\mathbb{R}^m$ given by the rule

(1) $$M_B\phi := \int_B \phi\circ P , \quad \text{all } \phi \in D(\mathbb{R}^m) .$$

Here, $P:\mathbb{R}^n \longrightarrow \mathbb{R}^m : x \longmapsto (x(i))_1^m$ is the canonical projection and $\int_K$ denotes the k-dimensional integral over K in case $\dim K = k$, i.e., K spans a k-dimensional flat.

It is obvious that M_B is nonnegative, with $\operatorname{supp} M_B \subseteq P[B]$. It is easy to see that M_B is a piecewise polynomial function of degree $\le n-m$ once one knows the recurrence relation $[BH_1]$

(2) $$D_{Pz}M_B = -\Sigma_i \langle z|n_i\rangle M_{B_i} , \quad \text{all } z \in \mathbb{R}^n .$$

Here,

$$D_y f := \Sigma\, y(i)\, D_i f ,$$

with $D_i f$ the partial derivative of f with respect to its i-th argument. Further, B_i denotes the typical (n-1)-dimensional polyhedron of which the boundary of B consists, and n_i denotes the corresponding outward normal. Finally, $\langle\cdot|\cdot\rangle$ denotes the scalar product.

In principle, M_B can be evaluated with the aid of the stable recurrence $[BH_1]$

(3) $$(n-m)M_B(Pz) = \Sigma_i \langle b_i - z|n_i\rangle M_{B_i}(Pz) , \quad \text{all } z \in \mathbb{R}^n ,$$

with b_i an arbitrary point in the flat spanned by B_i.

Cases of particular interest are:

(i) the simplex spline, obtained when B is a simplex. These B-splines were introduced in [B] following up on [S] and have already been studied intensively, mostly by W. Dahmen and C. A. Micchelli $[M_{1-2}]$, $[D_{1-4}]$, $[DM_{1-3}]$, but also by Goodman & Lee [GL], Hakopian $[Hk_{1-3}]$, and by Höllig $[Hl_{1-2}]$.

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(ii) the truncated powers or cone splines, obtained when B is a proper polyhedral cone spanned by some basis for $\mathbb{R}^n$. These were introduced by Dahmen [D_2] and further studied by Dahmen and Micchelli in [DM_3]. For example, they show that, near an extreme point of its support, a simplex spline coincides with a truncated power.

(iii) the box spline, obtained when B is a parallelepiped. These splines were introduced in [BD] and are the object of study of the present note.

To be precise, a box spline is, by a definition slightly more general than the one in [BD], a distribution M_Ξ on $\mathbb{R}^m$ given by the rule

(4) $$M_\Xi : \phi \longmapsto \int_{[0,1]^r} \phi\Big(\sum_{i=1}^{r} \lambda(i)\xi_i\Big)\, d\lambda$$

for some sequence $\Xi := (\xi_i)_1^r$. If $\dim \langle\Xi\rangle = m$, then

$$M_\Xi = M_B / \mathrm{vol}_r B$$

with

$$B := \{\sum_{i=1}^{r} \lambda(i)\hat{\xi}_i : \lambda \in [0,1]^r\}$$

the parallelepiped spanned by some linearly independent sequence $(\hat{\xi}_i)_1^r$ in $\mathbb{R}^n$ for which $P\hat{\xi}_i = \xi_i$, all i. We would like, though, to consider M_Ξ also in case $\dim \langle\Xi\rangle < m$.

For this, we find it convenient to enlarge the above definition of the B-spline M_B by allowing P in (1) to be an arbitrary linear map on B into $\mathbb{R}^m$. Then (1) defines M_B as the P-shadow of B. One checks that this leaves the recurrence relations (2) and (3) unchanged (see Sect.1).

In these terms, the box spline M_Ξ defined by (4) is the P-shadow of the box $[0,1]^r$, with P the linear map given by

$$P\lambda := \sum_{i=1}^{r} \lambda(i)\xi_i .$$

Here is an outline of the paper. We discuss P-shadows in Section 1. In Section 2, we give some basic information about the box spline M_Ξ, such as its recurrence relations, its Fourier transform, and its relationship to the difference operator Δ_Ξ and to the truncated powers. We show in Section 3 that it is usually possible to make a partition of unity out of the box spline and certain of its translates in many ways. We use this fact in Section 4 to show that the box spline and its translates are usually globally linearly

dependent, thus destroying all hopes for stability or the existence of a set of dual linear functionals for such sets except in special circumstances. One such is discussed in [BH_2].

In the remaining sections, we consider the space

$$S_\Xi \; := \; \{ \Sigma_V \, a(j) M_j \; : \; a \in \mathbb{R}^V \}$$

with

$$M_j \; := \; M_\Xi(\cdot - j) \; , \quad \text{all} \quad j \in V := \mathbb{Z}^m \; ,$$

and under the assumption that $\Xi \subseteq V$ and that $\langle \Xi \rangle = \mathbb{R}^m$. In Section 5, we determine all polynomials in S_Ξ as well as the largest k for which all polynomials of (total) degree k or less are contained in S_Ξ. We use this information in Section 6 to construct a quasi-interpolant from S_Ξ and thereby to obtain statements about the degree of approximation obtainable from $S_{\Xi,h} := \{x \longmapsto f(x/h) : f \in S_\Xi\}$ as $h \longrightarrow 0$.

We could have obtained our results concerning S_Ξ with the aid of the general theory of spaces spanned by translates of a fixed function developped by Fix and Strang [FS], particularly if we had been content to discuss only $\mathbb{L}_2$. We chose to derive our results directly since it seems no more effort to do this than it is to verify that the general conditions given in [FS] are satisfied for our specific examples.

We point out in Section 2 that $S_\Xi \subseteq \mathbb{L}_\infty^{(d)}$, with

$$d \; := \; \max \{ \; r \; : \; \langle \Xi \setminus Z \rangle = \mathbb{R}^m \;\; \text{for all} \;\; Z \subseteq \Xi \;\; \text{with} \;\; |Z| = r \; \}$$

(see Section 2 for how we treat the <u>sequence</u> Ξ as a set). This raises the question of the relationship of S_Ξ to the space of all pp functions in $\mathbb{L}_\infty^{(d)}$ on the same mesh and of degree $\leq |\Xi| - m$. We study this difficult question in [BH_2] just for $m = 2$ and mainly only for the 3-direction mesh, i.e., for $\operatorname{ran} \Xi = \{e_1, e_2, e_1+e_2\}$.

Notation. With $A \subseteq \mathbb{R}^m$, we denote by $[A]$ the convex hull of A and by $\langle A \rangle$ its linear span. We use $x(r)$ for the r-th entry of the vector x. For $x \in \mathbb{R}^m$ and $j \in \mathbb{Z}_+^m$, the number x^j is computed as

$$x^j \; := \; x(1)^{j(1)} \ldots x(m)^{j(m)},$$

as usual. We denote by π the class of all polynomials (on $\mathbb{R}^m$), and by π_k its subspace made up of those of total degree no larger than k. Thus

$$\pi_k \; := \; \{x \longmapsto \sum_{|j| \leq k} a(j) x^j\}$$

with $|j| := j(1) + \ldots + j(m)$. We also use $D^j := D_1^{j(1)} \ldots D_m^{j(m)}$ and, more generally, $p(D) := \Sigma_k\, a(k)D^k$ in case $p : x \longmapsto \Sigma_k a(k)x^k$. Here, we use again the notation $D_i f$ for the partial derivative with respect to its i-th argument of the function f with domain in $\mathbb{R}^m$. We also use the notation $D_y := \Sigma_1^m y(i)D_i f$. For a sequence of vectors in $\mathbb{R}^m$, such as $\Xi = (\xi_1,\ldots,\xi_r)$, we use

$$D_\Xi := D_{\xi_1} \ldots D_{\xi_r} .$$

We also use $\Delta_\Xi := \Delta_{\xi_1} \ldots \Delta_{\xi_r}$ and $\nabla_\Xi := \nabla_{\xi_1} \ldots \nabla_{\xi_r}$, with

$$\Delta_y f := f(\cdot+y) - f\ , \quad \nabla_y f := f - f(\cdot-y) \ .$$

Finally, we denote by $\mathcal{D}(\mathbb{R}^m)$ the space of tempered distributions on $\mathbb{R}^m$.

1. P-shadows. As defined in the introduction, the P-shadow of a convex polyhedron B in $\mathbb{R}^n$ is the distribution M on $\mathbb{R}^m$ given by the rule

$$M : \phi \longmapsto \int_B \phi \circ P \ , \quad \text{all} \quad \phi \in D(\mathbb{R}^m) \ ,$$

with $P:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ an affine map.

We claim that the recurrence relations for B-splines established in [BH_1;Theorem 2] remain valid for these more general B-splines and state this in the following theorem, for the record. For this, we make the assumption that P is a linear map, i.e., $P0 = 0$. This can always be achieved by a translation in $\mathbb{R}^n$. Further, we assume that B is proper, i.e., n-dimensional. If $r := \dim B < n$, then this can be achieved by restricting P to the affine hull of B and identifying this hull with $\mathbb{R}^r$. Given that B is a proper convex polyhedron, its boundary is made up of (n-1)-dimensional convex polyhedra B_i, with corresponding outward normals n_i, and b_i denotes an arbitrary point in the affine hull of B_i. Further, D stands for the first order differential operator given by the rule

$$Df := \sum_{i=1}^{r} \phi_i D_i f$$

in case f has its domain in $\mathbb{R}^r$, with

$$(\phi_i f)(x) := x(i)f(x) \ .$$

Thus $(Df)(x) = (D_x f)(x)$ and the adjoint of D is $-\Sigma\, D_i \phi_i$.

Theorem 1. Let B be a proper convex polyhedron in $\mathbb{R}^n$ and let M be its P-shadow in $\mathbb{R}^m$ under the linear map $P:\mathbb{R}^n \longrightarrow \mathbb{R}^m$. Then

(i) $D_{Pz}M = -\Sigma_i \langle z|n_i\rangle M_i$, all $z \in \mathbb{R}^n$.

(ii) $(n-m)M(Pz) = \Sigma_i \langle b_i - z|n_i\rangle M_i(Pz)$, all $z \in \mathbb{R}^n$.

(iii) $DM = (n-m)M - \Sigma_i \langle b_i|n_i\rangle M_i$.

Proof. The proof is a slight extension of the arguments for Theorem 2 of [BH_1]. The proof of (i) only needs the additional observation that

(1) $$D_y(\phi \circ P) = (D_{Py}\phi)\circ P \ .$$

This also implies that

(2) $$(D\phi)(Px) = (D_{Px}\phi)(Px) = (D_x(\phi \circ P))(x) = (D(\phi \circ P))(x) \ ,$$

of use in a moment. The recurrence relation (ii) follows from (i) and (iii). As to (iii), observe that

$$D_j\Phi_j = 1 + \Phi_j D_j ,$$

therefore

$$-(DM)\phi = \int_B \Big(\sum_{j=1}^{m} D_j\Phi_j\phi\Big)\circ P = mM\phi + \int_B (D\phi)\circ P$$

and

$$\int_B \sum_{i=1}^{n} D_i\Phi_i(\phi\circ P) = nM\phi + \int_B D(\phi\circ P) .$$

Here, the last integral in the first line equals the last integral in the second, by (2).

Thus

$$(DM)\phi = (n-m)M\phi - \sum_{i=1}^{n}\int_B D_i\Phi_i(\phi\circ P) ,$$

and the argument now finishes as in [BH_1]. |||

2. Basic properties of the box spline. The box spline M_Ξ defined in (0.4) (with $r = n$) is a <u>symmetric</u> function of the sequence $\Xi = (\xi_i)_1^n$. In other words, $M_{\Xi'} = M_\Xi$ for any rearrangement Ξ' of the sequence Ξ. For this reason, we find it excusable and, in any case, convenient to treat Ξ in the sequel as if it were a <u>set, of cardinality</u> n, rather than a sequence, even though the set $\{\xi_i : i=1,\dots,n\}$ may well have fewer than n elements. Thus, we write

$$\sum_{\xi\in\Xi} \lambda(\xi)\xi \quad \text{instead of} \quad \sum_{i=1}^{n} \lambda(i)\xi_i$$

or

$$\Xi\setminus\xi \quad \text{instead of} \quad \left(\xi_{s(i)}\right)_1^{n-1}$$

for the appropriate subsequence $s(1), \dots, s(n-1)$ of $1, \dots, n$. In the latter example, this abuse of notation stresses the fact that it doesn't matter which one of the possibly several occurrences of the vector ξ in the sequence Ξ is being omitted.

It is clear that M_Ξ is nonnegative and that

$$\text{(1)} \qquad \operatorname{supp} M_\Xi = \{\, \Sigma\lambda(\xi)\xi : \lambda \in [0,1]^\Xi \} .$$

Further, from (0.4),

$$\text{(2)} \qquad \|M_\Xi\| = 1$$

as a linear functional on $C(\mathbb{R}^m)$. Also,

$$\text{(3)} \qquad M_\Xi \in L_\infty \quad \text{iff} \quad \langle\Xi\rangle = \mathbb{R}^m .$$

The **recurrence relations** of Theorem 1 for general B-splines simplify for the box spline as follows.

Proposition 2.1. <u>If</u> $z = \Sigma_\Xi \lambda(\xi)\xi$, <u>then</u>

$$\text{(4)} \qquad D_z M_\Xi = \sum_{\xi\in\Xi} \lambda(\xi)\left(M_{\Xi\setminus\xi} - M_{\Xi\setminus\xi}(\cdot-\xi)\right) ,$$

$$\text{(5)} \qquad (n-m)M_\Xi(z) = \sum_{\xi\in\Xi} \left(\lambda(\xi)M_{\Xi\setminus\xi}(z) + (1-\lambda(\xi))M_{\Xi\setminus\xi}(z-\xi)\right) .$$

Proof. The typical facet (i.e., $(n-1)$-dimensional face) of $B = [0,1]^n$ has the form

$$B_\xi := \{\mu \in [0,1]^\Xi : \mu(\xi) = 0\}$$

or else the form $e_\xi + B_\xi$, for some $\xi \in \Xi$. Further, B_ξ and $e_\xi + B_\xi$ have the outward

normal $-e_\xi$ and e_ξ , respectively. Thus

$$\langle\mu|n_i\rangle = \begin{cases} -\mu(\xi) , & B_i = B_\xi \\ \mu(\xi) , & B_i = e_\xi + B_\xi \end{cases} ,$$

and (4) and (5) now follow from Theorem 1.(i) and (ii) . |||

Smoothness. We associate with Ξ the number

$$(6) \qquad d := \max \{ r : \langle\Xi\setminus Z\rangle = \mathbb{R}^m \text{ for all } Z \subseteq \Xi \text{ with } |Z| = r \}$$

and say for short that Ξ is **d-spanning.** (We take $d = -1$ in case $\langle\Xi\rangle \neq \mathbb{R}^m$). The number d is of interest here since it follows from (4) and (3) that all r-th order derivatives of M_Ξ are in L_∞ as long as $\langle\Xi\setminus Z\rangle = \mathbb{R}^m$ for all $Z \subseteq \Xi$ with $|Z| = r$. Thus

$$M_\Xi \in L_\infty^{(d)} \subseteq C^{(d-1)} .$$

Obviously, d cannot be bigger than $|\Xi| - m$ which is the total degree of the polynomial pieces of which M_Ξ consists. Precisely, on each connected component of the complement of

$$\{ [\Xi\setminus Z] + \sum_H \eta : H \subseteq Z , \langle\Xi\setminus Z\rangle \neq \mathbb{R}^m \} ,$$

M_Ξ agrees with some polynomial of degree $\leq |\Xi| - m$.

Examples. (i) For $m = 1$ and $\xi = e_1$, all $\xi \in \Xi$, M_Ξ is just the forward cardinal B-spline, i.e., $M_\Xi = M(\cdot;0,1,\ldots,n)$. For $m > 1$ and Ξ containing only $e_1, \ldots, e_m$ (each at least once), M_Ξ is the tensor product of such univariate B-splines.

(ii) For $m = 2$ and $|\Xi| = n = 3$, with $d = 1$, we obtain a standard linear finite element.

(iii) For $m = 2$ and $\Xi = (e_1, e_2, e_1+e_2, e_1-e_2)$, M_Ξ is a piecewise quadratic function first studied by Zwart [Z] and independently derived by Powell [P] and Sabin [PS]. Its support is shown in Figure 1 together with its "mesh", i.e., its lines of transition from one polynomial piece to a neighboring piece. The dotted mesh lines occur in the above references. Our construction makes clear that they do not appear in actuality since they do not lie in some one-dimensional image $P[F]$ of some face F of $[0,1]^n$.

(iv) Further examples for Ξ containing only e_1, e_2, e_1+e_2 and/or e_1-e_2 can be found in Sablonniere's study [S1] of smooth finitely supported pp functions on regular

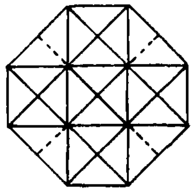

Figure 1. Support and meshlines for a C^1-quadratic box spline

meshes. The *"generalized triangular splines" of Frederickson* $[F_{1-2}]$ *can now be recognized* as spanned by the box spline M_Ξ, with Ξ containing each of the three vectors e_1, e_2, and e_3 the same number of times.

Associated difference operator. It follows from (4) that

$$D_\xi M_\Xi = M_{\Xi\backslash\xi} - M_{\Xi\backslash\xi}(\cdot-\xi) = \nabla_\xi M_{\Xi\backslash\xi},$$

therefore

(7) $$D_Z M_\Xi = \nabla_Z M_{\Xi\backslash Z}, \quad \text{for } Z \subseteq \Xi .$$

In particular,

$$D_\Xi M_\Xi = \nabla_\Xi \delta ,$$

since $M_\emptyset = \delta$:= point evaluation at 0 . Therefore

(8) $$\int M_\Xi D_\Xi \phi = (\Delta_\Xi \phi)(0), \quad \text{for all } \phi \in C^{|\Xi|}(\mathbb{R}^m) .$$

This close association between the box spline and the forward difference operator brings to mind the well known association

$$\int M(x;t_0,\ldots,t_n)\, \phi^{(n)}(x)/n!\, dx = [t_0,\ldots,t_n]\phi$$

between the univariate B-spline and the divided difference.

The **Fourier transform** of M_Ξ is quite simple,

(9) $$\hat{M}_\Xi(x) = \prod_{\xi\in\Xi} \frac{1 - e^{-i\xi\cdot x}}{i\xi\cdot x} .$$

From this we see that

(10) $$M_\Xi * M_Z = M_{\Xi\cup Z} .$$

Symmetries and local structure. We pointed out earlier that M_Ξ does not depend on the order of the vectors in the sequence Ξ . This is due to the fact that any linear map in $\mathbb{R}^n$ which permutes the unit vectors leaves the box $[0,1]^n$ invariant. Multiplication of some of the unit vectors by -1 will change $[0,1]^n$, but $[0,1]^n$ can be restored by a subsequent shift. Therefore

(11) $$M_{\Xi_\sigma} = M_\Xi(\cdot - \sum_{\xi\in\Xi} \frac{\sigma(\xi)-1}{2}\xi)$$

in case Ξ_σ is obtained from Ξ by multiplying each $\xi \in \Xi$ by $\sigma(\xi) \in \{-1,1\}$. A symmetry of a different sort occurs when Ξ' is the image of Ξ under some invertible linear map Q on $\mathbb{R}^m$. Precisely,

(12) $$M_\Xi = |\det Q|\, M_{Q\Xi} \circ Q .$$

This implies certain symmetries for M_Ξ in case $\Xi = Q\Xi$.

The box spline is particularly simple near an extreme point of its support.

Proposition 2.2. If e is an extreme point of $\operatorname{supp} M_\Xi$, then, in a neighborhood of e , M_Ξ agrees with some truncated power of degree $|\Xi| - m$. In particular, $M_\Xi(\cdot + e)$ is homogeneous of degree $|\Xi| - m$ near 0 .

Proof. We pointed out earlier that

$$\operatorname{supp} M_\Xi = \{\sum_{\xi\in\Xi} \lambda_\xi \xi : \lambda\in[0,1]^\Xi\} .$$

Thus any extreme point e of the (closed) support is necessarily of the form

$$e = \sum_{\zeta\in Z} \zeta$$

for some $Z \subseteq \Xi$ for which, further, there exists $\eta \in \mathbb{R}^m$ so that $\langle\eta|\xi\rangle > 0$ for all $\xi \in \Xi_\sigma$ with

$$\sigma(\xi) := \begin{cases} -1 , & \xi\in Z \\ 1 , & \xi\in\Xi\setminus Z \end{cases} .$$

Therefore, from (11),

$$M_\Xi(\cdot+e) = M_{\Xi_\sigma} ,$$

showing that it is sufficient to consider the case that $e = 0$ and, for some $\eta \in \mathbb{R}^m$ and all $\xi \in \Xi$, $\langle\eta|\xi\rangle > 0$.

In this case,

$$\varepsilon := \operatorname{dist}(0, [\Xi]) > 0 .$$

Therefore, for all test functions ϕ with $\operatorname{supp} \phi \subseteq B_\varepsilon(0)$:= ball of radius ε and center 0 ,

$$M_\Xi\phi = \int_{[0,1]^n} \phi\big(\sum \lambda(\xi)\xi\big)\, d\lambda = \int_{\mathbb{R}^n_+} \phi\big(\sum \lambda(\xi)\xi\big)\, d\lambda \quad . |||$$

3. Partition of unity. In this section, we show that appropriate translates of an appropriately scaled version of the box spline

$$M := M_{\Xi}$$

form a partition of unity. By a standard argument, this implies that the space spanned by these translates can, at least, approximate continuous functions (as the mesh size is reduced by scaling).

Proposition 3. <u>Suppose</u> Ξ <u>contains the basis</u> Z <u>(for</u> $\mathbb{R}^m$ <u>). Then</u>

(1)
$$\sum_{j\in\mathbb{Z}^Z} M\big(\cdot - \sum_{\zeta\in Z} j(\zeta)\zeta\big) = 1/|\det Z| .$$

Proof. Since $\mathbb{R}^m$ is the essentially disjoint union of the sets

$$\{ \sum_{\zeta\in Z} \big(\lambda(\zeta)-j(\zeta)\big)\zeta : \lambda \in [0,1]^Z\} , \quad j \in \mathbb{Z}^Z ,$$

we find that

$$\sum_{j\in\mathbb{Z}^Z} M\big(\cdot - \Sigma_Z j(\zeta)\zeta\big)\phi = \int_{[0,1]^{n-m}} \Big(\int_{\mathbb{R}^m} \phi\big(\Sigma_Z \lambda(\zeta)\zeta + \Sigma_{\Xi\backslash Z}\mu(\xi)\xi\big) \, d\lambda\Big) \, d\mu .$$

The change of variables $\lambda \longmapsto \Sigma_Z\lambda(\zeta)\zeta$ carries this to

$$\int_{[0,1]^{n-m}} \Big(\int_{\mathbb{R}^m} \phi\big(x + \Sigma_{\Xi\backslash Z}\mu(\zeta)\zeta \big) \, dx/|\det Z| \Big) \, d\mu$$

and this equals

$$\int_{\mathbb{R}^m} \phi \; \mathrm{vol}_{n-m}[0,1]^{n-m}/|\det Z| . \quad |||$$

Corollary. <u>If</u> $\Xi \subseteq \mathbb{Z}^m$ <u>and</u> $\langle\Xi\rangle = \mathbb{R}^m$, <u>then</u> $\sum_{j\in\mathbb{Z}^m} M(\cdot-j) = 1$.

Proof. Let $Z \subseteq \Xi$ be a basis (for $\mathbb{R}^m$). Then

$$A := \{\Sigma_Z \; j(\zeta)\zeta : j \in \mathbb{Z}^Z\}$$

is a subgroup of $\mathbb{Z}^m$ and its factor group $G := \mathbb{Z}^m/A$ has (finite) order $|\det Z|$. Therefore

$$\sum_{j\in\mathbb{Z}^m} M(\cdot-j) = \sum_{g\in G} \sum_{j\in A} M(\cdot-g-j) = |G|/|\det Z| = 1 . \quad |||$$

4. Linear independence of translates. For any particular subset V of $\mathbb{R}^m$, we consider the collection of translates $M_v := M(\cdot - v)$, $v \in V$, of the box spline

$$M := M_{\Xi}.$$

Such a collection is always (algebraically) linearly independent: Indeed, if $f := \Sigma_V a(v) M_v$ with $W := \operatorname{supp} a$ a finite nonempty set, then

$$(\operatorname{supp} M_w) \backslash \bigcup_{v \in W \backslash w} (\operatorname{supp} M_v) \neq \emptyset$$

for some $w \in W$, hence $f \neq 0$.

We are interested in considering nontrivial sums of infinitely many translates. For this, we make the assumption that V has no finite limit points. Then only finitely many of the translates have any particular point in their support and thus, for $a \in \mathbb{R}^V$ with suitably controlled growth at infinity,

$$f := \Sigma_V a(v) M_v$$

defines a distribution on $\mathbb{R}^m$.

Assume that M is a function, i.e., that Ξ contains a basis (for $\mathbb{R}^m$). Then

$$S_{\Xi,V} := \operatorname{span}\,(M_v)_V := \{\Sigma_V a(v) M_v : a \in \mathbb{R}^V\}$$

is a space of piecewise polynomial functions, possibly quite smooth, and it becomes of interest to find out to what an extent $(M_v)_V$ is a basis for this space or one of its subspaces. We call $(M_v)_V$ **(globally) linearly independent** if the linear map

$$a \longmapsto \Sigma_V a(v) M_v \tag{1}$$

is 1-1 on $\mathbb{R}^V$. Such linear independence is a first necessary condition for other properties of interest to hold. One such property is **stability** of the basis $(M_v)_V$ for $S_{\Xi,V}$, i.e., the property that the map (1) is bounded and bounded below on $\ell_\infty(V)$ into $L_\infty(\mathbb{R}^m)$. Another is the possibility of interpolation from $S_{\Xi,V}$, i.e., the existence of points p_v, typically with $M_v(p_v) \neq 0$, all $v \in V$, so that, for any function f in some class K, there exists one and only one $s \in S_{\Xi,V} \cap K$ which agrees with f at $(p_v)_V$. We make clear below that this (global) linear independence is usually not present. Yet, as is pointed out in [BD], if $\Sigma_V a(v) M_v = 0$ and $a \neq 0$, then there exists $r > 0$ so that, for all $v \in V$, a changes sign on $V \cap B_r(v)$. This implies that the map (1) is 1-1 on $\pi_{|V}$.

The space $S_{\Xi,V} = \text{span}\,(M_v)_V$ becomes interesting when V is related to Ξ. By assumption, M is a function, i.e., Ξ contains a basis Z (for $\mathbb{R}^m$). Therefore, according to Proposition 3.1, the collection M_v, $v \in V_Z := \{ \Sigma_Z\, j(\zeta)\zeta : j \in \mathbb{Z}^Z\}$ forms a partition of the constant $1/|\det Z|$. This suggests consideration of V of the form V_Z for some basis Z in Ξ. We go one step further, though, and consider from now on only the following <u>normalized</u> situation:

(2) $$\Xi \subseteq V = \mathbb{Z}^m .$$

This is the same, up to an affine transformation, as the assumption that $V_Z \subseteq V$ for all bases Z in Ξ. We abbreviate

$$S_\Xi := S_{\Xi,\mathbb{Z}^m} .$$

Proposition 4. <u>Under the assumption (2),</u> $(M_j)_V$ <u>is linearly dependent unless</u>

$$|\det Z| = 1 \quad \underline{\text{for all bases}} \quad Z \subseteq \Xi .$$

Proof. By assumption, Ξ contains a basis Z for $\mathbb{R}^m$, therefore $(|\det Z| M_j)_{V_Z}$ provides a partition of unity as does $(M_j)_V$, by Proposition 3 and its corollary. If now $|\det Z| \neq 1$ for some basis Z in Ξ, then $V_Z \neq V$, yet

$$\Sigma_{V_Z} |\det Z|\, M_j = 1 = \Sigma_V\, M_j . \quad |||$$

Remark. It would be nice to know whether the converse of this proposition holds.

5. The polynomials in S_Ξ . In this section, we determine $S_\Xi \cap \pi$. This information is important in the discussion of the degree of approximation to smooth functions attainable from $S_{\Xi,h}$. We continue to use the abbreviations and assumptions introduced in Section 4.

Lemma 5.1. $\pi \cap \ker D_\Xi = \pi \cap \ker \Delta_\Xi$.

Proof. Recall from (2.8) that

$$\bigl(\Delta_\Xi f\bigr)(x) = \int M\, D_\Xi f(\cdot + x) .$$

Therefore $\pi \cap \ker \Delta_\Xi \supseteq \pi \cap \ker D_\Xi$. For the converse, observe that $\Delta_\Xi f = 0$ implies $\int M(\cdot - x)\, D_\Xi f = 0$ for all x . Since M is nonnegative and of compact support, this cannot hold for a polynomial f unless the polynomial $D_\Xi f$ vanishes identically. |||

We also note that (2.7) together with summation by parts gives

(1) $$D_Z\bigl(\Sigma\, a(j)\, M_j\bigr) = \Sigma\, \bigl(\nabla_Z a\bigr)(j)\, M_{\Xi\setminus Z}(\cdot - j) , \quad \text{all } Z \subseteq \Xi .$$

Theorem 5. Let $K := \bigcap_{Z \in \Xi^*} \ker D_Z$ with $\Xi^* := \{Z \subseteq \Xi : \langle \Xi \setminus Z\rangle \neq \mathbb{R}^m\}$. Then

(2) $$\pi \cap S_\Xi = \pi \cap K .$$

Proof. Let

$$\Sigma\, a(j)\, M_j =: p \in \pi_k \cap S_\Xi .$$

If $Z \subseteq \Xi$, then, by (1), the polynomial $D_Z p$ can be written

$$D_Z p = \Sigma\, \bigl(\nabla_Z a\bigr)(j)\, M_{\Xi\setminus Z}(\cdot - j) .$$

If also $\langle \Xi \setminus Z\rangle \neq \mathbb{R}^m$, then $\operatorname{supp} M_{\Xi\setminus Z}$ has zero measure, hence the polynomial $D_Z p$ must vanish identically.

For the converse statement, we prove by induction on k that

(3) $$S_\Xi \supseteq \pi_k \cap K ,$$

it being trivially true for $k = -1$. For the induction step, we show now that

(4) $$p \in \pi_k \cap K \quad \text{implies} \quad q := p - \Sigma\, p(j) M_j \in \pi_{k-1} \cap K\,.$$

Since p belongs to K, so does $\Sigma\, p(j) M_j$, by Lemma 5.1 and (1) (making use of the fact that $\ker \Delta_\Xi = \ker \nabla_\Xi$). Thus it remains to show that $q \in \pi_{k-1}$. This is established once we show that, for any $y \in \mathbb{R}^m$,

$$(D_y)^k q = 0\,.$$

For this, we note that, whenever $Z \not\in \Xi^*$, then

$$(D_y)^s D_Z = (D_y)^{s-1} \sum_{\xi \in \Xi \setminus Z} a(\xi)\, D_{Z\setminus\xi}$$

(with $y = \Sigma_{\xi\in\Xi\setminus Z} a(\xi)\xi$). Repeated application of this formula justifies the claim that

$$(D_y)^k = \sum_{Z\in\Xi^*, |Z|<k} a(Z)(D_y)^{k-|Z|} D_Z + \sum_{Z\in\Xi, Z\not\in\Xi^*, |Z|=k} a(Z) D_Z\,.$$

It follows that

$$(D_y)^k q = \sum_{Z\in\Xi, Z\not\in\Xi^*, |Z|=k} a(Z)\big(D_Z p - \Sigma(\nabla_Z p)(j) M_{\Xi\setminus Z}(\cdot - j)\big)$$

and this is zero since, for each $Z \subseteq \Xi$ with $Z \not\in \Xi^*$, we have $\Sigma\, M_{\Xi\setminus Z}(\cdot - j) = 1$ by the corollary to Proposition 3, while $|Z| = k$ implies that $\nabla_Z p = D_Z p$ is some constant, since $p \in \pi_k$.

It is now easy to complete the induction step. If $p \in \pi_k \cap K$, then, by (4), $p \in S_\Xi + \pi_{k-1} \cap K$, *hence* $p \in S_\Xi$ *by induction hypothesis.* |||

Corollary 1. For each k, the map $T: p \longmapsto \Sigma\, p(j)\, M_j$ carries $\pi_k \cap S_\Xi$ 1-1 onto itself.

Proof. We mentioned already in Section 4 that T is 1-1 on π. Thus it is sufficient to show that T carries $\pi_k \cap S_\Xi$ into itself. But that is obvious since, by (4), even $(1-T)[\pi_k \cap S_\Xi] \subseteq \pi_{k-1} \cap S_\Xi$. |||

Corollary 2. As in (2.6), let

$$d := \max\{r : \langle \Xi \setminus Z \rangle = \mathbb{R}^m \text{ for all } Z \subseteq \Xi \text{ with } |Z| = r\}\,.$$

Then $\pi_k \subseteq S_\Xi$ if and only if $k \le d$.

Proof. From the theorem

$$\pi_k \subseteq S_\Xi \quad \text{iff} \quad \pi_k \subseteq K := \bigcap_{Z \in \Xi^*} \ker D_Z .$$

Further, the differential operator D_Z decreases the total degree of any polynomial by at least $|Z|$ and, for some polynomials, by exactly $|Z|$. Finally,

$$d+1 = \min_{Z \in \Xi^*} |Z| .$$

This shows that $\pi_d \subseteq \ker D_Z$ for all $Z \in \Xi^*$. It also shows that, for some $Z \in \Xi^*$, $|Z| = d+1$, hence $D_Z p \neq 0$ for some $p \in \pi_{d+1}$. |||

6. Degree of approximation from S_Ξ **.** In this last section, we discuss the degree of approximation from $S_{\Xi,h}$ to a sufficiently smooth function, as $h \longrightarrow 0$. Here, h indicates a scaling of the mesh, i.e.,

$$S_{\Xi,h} \;:=\; \{x \longmapsto f(x/h) \,:\, f \in S_\Xi\} \,.$$

Theorem 6. <u>If</u> $k \le d$ (<u>with</u> d <u>given by (2.6)), then there exists a linear functional</u> λ <u>on</u> π_k <u>so that</u> $p = \Sigma_j\, \lambda p(\cdot+j)\, M_j$ <u>for all</u> $p \in \pi_k$.

Proof. By Corollary 2 of Theorem 5, $\pi_k \subseteq S_\Xi$, while, by Corollary 1 of Theorem 5, the map $T{:}p \longmapsto \Sigma\, p(j)\, M_j$ is 1-1 onto π_k . Thus

$$p \;=\; \Sigma_j \big(T^{-1}p\big)(j)\, M_j \;,\quad \text{all}\quad p \in \pi_k \,,$$

with $T^{-1} := \big(T_{|\pi_k}\big)^{-1}$. Further, with T_i the shift by the vector i , i.e.,

$$\big(T_i p\big)(x) \;:=\; p(x+i) \;,\quad \text{all } x \;,$$

we have

$$\Sigma\, \big(T_i p\big)(j) M_j \;=\; \Sigma\, p(j+i) M_j \;=\; \Sigma\, p(j) M_j(\cdot+i) \;=\; T_i\big(\Sigma\, p(j) M_j\big) \,,$$

showing that T commutes with T_i , hence so does T^{-1} . This proves the theorem, with $\lambda p \;:=\; \big(T^{-1}p\big)(0)$. |||

The theorem implies statements about degree of approximation to smooth functions from S_Ξ in the now standard quasi-interpolant fashion: Let K be the class of functions which belong locally to some function space K_0 , e.g., to L_1 . Extend the linear functional λ of the theorem to a continuous linear functional μ on K_0 and with support in supp M . The quasi-interpolant scheme

$$Q{:}K \longrightarrow S_\Xi{:}\; f \longmapsto \Sigma\, \mu f(\cdot+j) M_j$$

then reproduces π_k and is local. This implies that

$$f - Qf \;=\; (f-p) - Q(f-p) \;,\; \text{for all}\;\; p \in \pi_k$$

and that

$$|(Qg)(x)| \;=\; |\Sigma \mu g(\cdot+j) M_j(x)| \;\le\; \|\mu\| \max\{ \|g_{|\mathrm{supp} M_j}\| : M_j(x) \ne 0\} \;.$$

Therefore

$$|(f - Qf)(x)| \le \mathrm{dist}_\rho(f, \pi_k)$$

with

$$\rho(g) := |g(x)| + \|\mu\| \max\{\|g_{|\mathrm{supp} M_j}\| : M_j(x) \ne 0\} .$$

This shows that Qf approximates to f locally as well as local polynomial approximation. Here is a particular result along these lines.

Corollary. Let μ be an extension of λ to a continuous linear functional on $L_\infty(\mathrm{supp}\ M)$ and let $Q_h := S_h Q S_{1/h}$ with $Q: f \longmapsto \Sigma\ \mu f(\cdot+j) M_j$ and $(S_h f)(x) := f(xh)$. Then, for $f \in L_\infty^{(k+1)}$, $\|f - Q_h f\|_\infty = O(h^{k+1})$.

Sharpness. The order $O(h^{d+1})$ is, in general, best possible for the approximation from S_Ξ to smooth functions. To see this, choose $Z \in \Xi^*$ (cf. Theorem 5) with $|Z| = d+1$ and a polynomial $p \in \pi_{d+1}$ with $D_Z p = 1$. If, for some approximating sequence $s_h \in S_{\Xi,h}$, we have

$$\|s_h - p\|_{L_1[0,1]^m} = o(h^{d+1}) ,$$

it follows from the standard Markov inequality for piecewise polynomials that

$$\|D_{Z'} s_h - D_{Z'} p\|_{L_1[0,1]^m} = o(h) \tag{1}$$

for any $Z' \subseteq Z$ with $|Z'| = d$. Set $z = Z \setminus Z'$. Since the support of $D_z(D_{Z'} s_h)$ is contained in hyperplanes, $(D_{Z'} s_h)$ is piecewise constant on lines in the direction z. On the other hand, $D_{Z'} p$ restricted to these lines is of the form $(D_{Z'} p)(x + tz) = (D_{Z'} p)(x) + t$ which contradicts (1). |||

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2320	2. GOVT ACCESSION NO. AD-A114479	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) B-SPLINES FROM PARALLELEPIPEDS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) C. de Boor and K. Höllig		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 NSF- MCS-7927062, Mod. 1
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis & Computer Science
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE February 1982
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

B-splines
multivariate
spline functions
degree of approximation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study multivariate B-splines M obtained as "shadows" of parallelepipeds, and spaces spanned by their translates $S = \operatorname{span}_{j \in \mathbb{Z}^m} M(\cdot - j)$. Recurrence relations for M are obtained and a necessary condition for the stability of the B-spline basis is given. We further determine the polynomials contained in S and the optimal degree of approximation from S.

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